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Brownian-motion ensembles: correlation functions of determinantal processes

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Abstract

We calculate the n-point correlation function for a large class of Brownian-motion ensembles of random matrix theory. The corresponding Fokker–Planck equation describes a determinantal process in the theory of matrix-valued stochastic differential equations and can be mapped onto a Schödinger equation of non-interacting electrons in one dimension. The correlation functions are obtained explicitly via a suitable generalization of the method of biorthogonal functions.

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1. Introduction

Brownian-motion ensembles (BME) are natural generalizations of the standard ergodic ensembles of random-matrix theory (RMT). The first formulation of a parametric random-matrix ensemble in terms of a Hermitian matrix-valued diffusion process appeared in the classical paper by Dyson [1]. More recently, much efforts [2–11] have been devoted to extend Dyson's Brownian-motion model to more general classes of stochastic processes, such as non-Hermitian processes (with unitary or pseudo-unitary matrices) and non-standard symmetry classes taken from Cartan's table of Riemannian symmetric spaces [12–14] (Bogoliubov-de Gennes and chiral). A central motivation for constructing a general classification scheme of RMT using the theory of stochastic processes is the possibility of combining non-perturbative tools from various different applications of RMT into a unified technique which may uncover hidden algebraic structures and novel physical phenomena on a given specific problem.

There are two approaches to build such a classification. The first one is based on stochastic differential equations and has been pursued vigorously by several authors [8–10] with emphasis on Hermitian matrix-valued processes. In a recent paper [11] we introduced an alternative approach based on the general algebraic structure of the Fokker–Planck equation.

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We have shown that for homogeneous processes with continuous sample paths a Fokker–Planck equation with 'time'-independent drift and diffusion coefficients can be used to model the parametric evolution of the joint distribution of level positions (the eigenvalues of certain random matrices). These 'non-equilibrium' ensembles of random-matrix theory are natural extensions of Dyson's Brownian-motion ensembles. Remarkably, their construction does not depend on the existence of an underlying matrix model, which in turn implies the validity of a classification scheme, alternative to Cartan's table of symmetric spaces, that is entirely based on algebraic properties of the Fokker–Planck operator. In this new scheme, Dyson's index β of standard RMT becomes a free parameter that can assume values different from the classical ones 1, 2 and 4. Extensions of RMT ensembles to arbitrary values of β have been part of current research topics in RMT and appeared in connection with Calogero–Sutherland models with rational couplings [15], β-ensembles [16] and free probability theory [17].

The central quantity in BME is the n-point correlation function defined as

$$\rho_n(x_1, \ldots, x_n; t) = \frac{N!}{(N-n)!} \int_a^b \mathrm{d}x_{n+1} \ldots \int_a^b \mathrm{d}x_N \, P(x_1, \ldots, x_N; t), \tag{1}$$

which represents the probability density of finding n levels around each of the positions $x_1, \ldots, x_n$ at 'time' t without observing the remaining ones. Knowledge of all n-point correlation functions is fundamental for a complete statistical description of the physical system of interest. For example, the average and variance of a linear statistics, i.e. a function of the form $F = \sum_{i=1}^{N} f(x_i)$, are given respectively by

$$\langle F \rangle = \int_a^b \mathrm{d}x \; f(x)\rho_1(x; t), \tag{2}$$

and

$$\mathrm{var}(F) = \int_a^b \int_a^b \mathrm{d}x \, \mathrm{d}y f(x) f(y) C(x, y; t), \tag{3}$$

in which we introduced the two-level function

$$C(x, y; t) = \delta(x - y)\rho_1(x; t) + \rho_2(x, y; t) - \rho_1(x; t)\rho_1(y; t). \tag{4}$$

Linear statistics play a central role in the theory of quantum transport, since they are related to many physical observables, such as the cumulants of the full counting statistics of charge transfer through the system [18].

For stationary ensembles the n-point correlation functions can be calculated explicitly by various methods [19]. The most remarkable feature is the ubiquity of a factorized structure, which allows the n-point function to be written as a conventional or quaternion determinant whose entries depend on a certain two-point function called the kernel. Such factorization is fundamental for the calculation of the statistical properties of physical observables and plays a role similar to Wick's theorem in many-body physics. For non-equilibrium ensembles these n-point correlation functions are much harder to calculate. For instance, in [11] we showed that the time evolution of the n-point correlation function (1), for a certain class of Brownian-motion ensembles, is given by the following hierarchy of singular integro-differential equations

$$\frac{\partial}{\partial t}\rho_n(x_1, \ldots, x_n; t) = \sum_{p=1}^{n} \left\{ \frac{\partial}{\partial x_p} \left(s(x_p) w_N J_\beta \frac{\partial}{\partial x_p} (w_N J_\beta)^{-1} \right) \rho_n(x_1, \ldots, x_n; t) \right.$$
$$\left. - \beta \frac{\partial}{\partial x_p} s(x_p) \mathcal{P} \int \frac{\rho_{n+1}(x_1, \ldots, x_n, x_{n+1}; t)}{x_p - x_{n+1}} \, \mathrm{d}x_{n+1} \right\}, \tag{5}$$

where $\mathcal{P}$ denotes the principal value of the integral and J_β, w_N and $s(x)$ will be defined in section 2. Thus to obtain exact expressions for these correlation functions we must, in principle, uncouple this hierarchy, which is a very non-trivial task indeed.

Exact solutions for non-equilibrium RMT ensembles have been, however, obtained for certain particular problems, such as the orthogonal-unitary and the symplectic-unitary crossover in Gaussian and circular ensembles [21–23], and for DMPK equations in the Wigner–Dyson unitary class [3], chiral unitary class [5] and in two types of BdG classes [6]. Remarkably, some of these solutions are intimately related to certain types of Harish–Chandra–Itzykson–Zuber (HCIZ) integrals [25], whose solutions are known only for connected and compact groups. In some cases one can make a direct use of the HCIZ integral for the unitary group to obtain explicit expressions for the n-point correlation function, as in the problem of a random Hermitian matrix coupled to a external matrix source [26–28]. Recently, Baker and Forrester [29] used Jack polynomials methods to derive explicit expressions for the transition probability of the Fokker–Planck equation describing Brownian motion in orthogonal polynomial ensembles (OPE), more specifically in the Hermite and Laguerre ensembles, for all symmetry classes (generic β). Despite the great importance of this result, the correlation function problem remains open, as it is not obvious how to obtain the n-point function from these author's representations.

In this work we employ the classification scheme introduced in [11] to unify the methods for calculating the n-point correlation function of Brownian-motion ensembles describing determinantal processes without directly uncoupling the hierarchy (5). Determinantal processes can be mapped onto a non-interacting many-body quantum problem, which allows the n-point function to be factorized as a determinant of a time-dependent kernel. Our approach is a generalization of the method of biorthogonal functions used in [5]. We derive a very useful integral representation for the kernel in terms of single particle Green's functions. In particular, for Hermite and Laguerre ensembles our results reproduce the formulae derived by Baker and Forrester. We also adapt our generalized biorthogonal functions method to work in the context of random transfer-matrix ensembles, which have been used in the description of electronic transport in disordered wires. The paper is organized as follows. In section 2 we present the Brownian-motion ensembles and describe the connection between the Fokker–Planck equation and the Schrödinger equation of the associated Calogero-Sutherland problem. In section 3 we restrict the problem to determinantal processes, for which the Schrödinger equation describes non-interacting particles, and present an exact solution of the Fokker–Planck equation with a symmetrized delta-function initial condition. In section 4 we introduce a generalization of the method of biorthogonal functions to calculate the n-point correlation function in the context of polynomial ensembles. Some examples are given in section 5, in which we apply such new method to the Hermite, Laguerre and Jacobi ensembles with degenerate initial conditions. We also show an interesting connection between the BME and the HCIZ integral in section 6. In section 7, we work out transfer-matrix ensembles by calculating the n-point correlation functions associated with certain DMPK equations. Finally, in the appendix we make contact with the Barker and Forrester's result.

2. Orthogonal polynomial Brownian motion ensembles

In [11] we studied orthogonal polynomial Brownian-motion ensembles, i.e., non-equilibrium ensembles in which the levels $\{x\} = (x_1, x_2, \ldots, x_N)$ perform a Brownian motion. The evolution equation of the joint level distribution $P(\{x\}, t)$ was constructed from the general theory of multivariate Markovian stochastic processes by imposing the following conditions: (i) path continuity; (ii) homogeneity; (iii) equilibrium distribution given by classical random matrix ensembles and (iv) a complete set of generalized multivariate classical polynomials as eigenfunctions. With these conditions we concluded that $P(\{x\}, t)$ satisfies the following

Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \mathcal{L}_{\mathrm{FP}} P, \tag{6}$$

where the Fokker–Planck operator is given by

$$\mathcal{L}_{\mathrm{FP}} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(J_\beta w_N s(x_i) \frac{\partial}{\partial x_i} \frac{1}{J_\beta w_N} \right), \tag{7}$$

with

$$J_\beta(\{x\}) = |\Delta_N(\{x\})|^\beta, \qquad \Delta_N(\{x\}) = \prod_{i<j} (x_i - x_j) \qquad \text{and} \qquad w_N(\{x\}) = \prod_{i=1}^{N} w(x_i). \tag{8}$$

The above stochastic process is constructed to make contact with the orthogonal polynomial ensembles by choosing $w(x)$ to be the weight function of such polynomials, $p_n(x)$, defined on the interval $[a, b]$. The auxiliary function $s(x)$ must be a polynomial with real roots of degree not greater than 2, i.e. $s(x) = s_0 + s_1 x + s_2 x^2$. These functions are not independent, but are chosen in such a way that their combination

$$\frac{1}{w(x)} \frac{\mathrm{d}}{\mathrm{d}x} (w(x) s(x)) = r(x) = r_0 + r_1 x, \tag{9}$$

is a polynomial of first degree[1], and must satisfy the boundary condition

$$w(a)s(a) = 0 = w(b)s(b). \tag{10}$$

Therefore, the stationary solution of the Fokker–Planck equation (6) and (7)

$$P_{st}(\{x\}) = C_N J_\beta(\{x\}) w_N(\{x\}) \tag{11}$$

reproduces the joint probability distribution of classical orthogonal polynomials ensembles, obtained in RMT from of a maximum entropy principle [19], with Dyson's index, β, classifying the ensembles as orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$). The normalization constant C_N can be obtained from the associated Selberg integral.

The Fokker–Planck equation (6) can also be written in the conventional form

$$\frac{\partial P}{\partial t} = \sum_{i=1}^{N} \left(-\frac{\partial}{\partial x_i} D_i^{(1)} + \frac{\partial^2}{\partial x_i^2} D_i^{(2)} \right) P, \tag{12}$$

with drift and diffusion coefficients defined respectively by

$$D_i^{(1)} = r(x_i) + \beta \sum_{j(\neq i)} \frac{s(x_i)}{x_i - x_j} \qquad \text{and} \qquad D_i^{(2)} = s(x_i). \tag{13}$$

We remark that the Brownian-motion ensembles depend solely on algebraic properties of the functions $w(x), s(x)$ and $r(x)$, and do not require a specific matrix realization. Furthermore, Dyson's index β appears as a free parameter and can assume values different from the classical ones $\{1, 2, 4\}$. The classification of the classical orthogonal polynomial ensembles put forward in [11] is summarized in table 1.

An interesting subject to study is the evolution of an arbitrary function of the levels, $F = F(x_1, \ldots, x_N)$, in a system whose dynamics is described by the Fokker–Planck equation (6). The ensemble average of such function evolves in time according to the equation

$$\langle F \rangle = \int \mathrm{d}^N x F(\{x\}) P(\{x\}, t). \tag{14}$$

[1] The coefficients r_0, r_1, s_0, s_1 and s_2 are real numbers.

Table 1. Classification of Brownian-motion ensembles based on properties of the functions $w(x)$ and $s(x)$, and its boundary conditions. In this unified scheme each symmetry class is described by its corresponding Fokker–Planck equation.

Interval	$w(x)$	$s(x)$	$r(x)$	Ensemble
$(-\infty, \infty)$	e^{-x^2}	1	$-2x$	Hermite
$[0, \infty)$	$x^{\nu}\mathrm{e}^{-x}$ $(\nu > -1)$	x	$1+\nu-x$	Laguerre
$[-1, 1]$	$(1-x)^{\nu}(1+x)^{\mu}$ $(\nu, \mu > -1)$	$1-x^2$	$\mu-\nu-(2+\mu+\nu)x$	Jacobi

Taking the time derivative of $\langle F\rangle$ we find the evolution equation

$$\frac{\mathrm{d}\langle F\rangle}{\mathrm{d}t} = \left\langle \mathcal{L}_{\mathrm{FP}}^{\dagger} F\right\rangle, \tag{15}$$

in which $\mathcal{L}_{\mathrm{FP}}^{\dagger}$ is the adjoint Fokker–Planck operator. Such equation-of-motion method was used in [11] to calculate ensemble average of transport observables, such as conductance and shot-noise power, in quantum dots. This method can also be used to obtain the evolution equation of the n-point correlation function. We start by noting that such function can be written as an ensemble average of a composition of delta functions

$$\rho(x_1, \ldots, x_n; t) = {\sum_{\{l\}}}' \left\langle \prod_{i=1}^{n} \delta\left(x_i - y_{l_i}\right)\right\rangle_y, \tag{16}$$

in which we introduced the composed index $\{l\} = l_1, l_2, \ldots, l_n$ with $l_i = 1, \ldots, N$. The prime indicates that the sum is over distinct indices and the subscript y to emphasize that the average is taken over these variables. By combining (16) with (15) we obtain the hierarchic equation (5). For a detailed deduction see [11]. In the following sections we will show how to calculate the n-point correlation function for determinantal process without directly uncoupling the hierarchy (5). We remark that since the results are exact they immediately satisfy (5).

2.1. Connection to Calogero–Sutherland quantum systems

Let us turn back to the non-equilibrium situation described by (6). One of the methods to solve Fokker–Planck equations is based on mapping the non-Hermitian operator $\mathcal{L}_{\mathrm{FP}}$ onto a Hermitian operator $\mathcal{H}$, which can be interpreted as a Hamiltonian of a quantum system [30]. This map is implemented by the Sutherland's similarity transformation

$$P(\{x\}, t) = w_N J_{\beta}^{1/2}\, \mathrm{e}^{-E_0 t}\, \Psi(\{x\}, t), \tag{17}$$

which maps the Fokker–Planck equation (7) onto a Schrödinger equation in imaginary time

$$-\frac{\partial \Psi}{\partial t} = \mathcal{H}\Psi. \tag{18}$$

Choosing the constant E_0 to be

$$E_0 = \beta N(N-1)(s_2\beta(N-2) + 3r_1)/12, \tag{19}$$

we can show [11] that $\mathcal{H}$ is a many-particle Hamiltonian operator of Calogero–Sutherland type

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{1}{w(x_i)} \frac{\partial}{\partial x_i}\left(w(x_i)s(x_i)\frac{\partial}{\partial x_i}\right) + \frac{\beta(\beta-2)}{4} \sum_{i\neq j} \frac{s(x_i)}{(x_i - x_j)^2}, \tag{20}$$

that describes N non-relativistic particles on a line interacting with a pairwise inverse-square potential. Since the levels execute a non-colliding Brownian motion, the joint distribution $P(\{x\}, t)$ vanishes at coincident points, thus allowing us to treat the particles in the associated quantum problem as fermions. The antisymmetry of the wavefunction is guaranteed by the following transformation

$$J_\beta^{1/2} \to J_\beta^{1/2} \mathrm{sign}\Delta_N(\{x\}). \tag{21}$$

This operator method is efficient if one can solve the associated Schrödinger eigenvalue problem. This approach is particularly tractable in the unitary case, $\beta = 2$, which corresponds to the non-interacting case. For general β the eigenfunctions of (20) are related to the multivariate orthogonal polynomials [11, 29], which in turn are connected to Jack symmetric polynomials. Such connection was explored by Baker and Forrester in [29] to construct a useful representation for the solution of the Fokker–Planck equation (7) for general values of β, which we shall discuss in appendix A.

2.2. *Formal solution: probability transitions and green's functions*

The general solution of the Fokker–Planck equation (6) can be written as

$$P(\{x\}, t) = \int \mathrm{d}^N x' \mathcal{P}(\{x\}, t|\{x'\}, 0) P(\{x'\}, 0), \tag{22}$$

where the transition probability satisfies the Fokker–Planck equation

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\mathrm{FP}}\right) \mathcal{P}(\{x\}, t|\{x'\}, 0) = 0 \tag{23}$$

with a symmetrized δ-function initial condition

$$\mathcal{P}(\{x\}, 0|\{x'\}, 0) = \frac{1}{N!} \sum_{P \in S_N} \prod_{j=1}^{N} \delta(x_j - x'_{P(j)}), \tag{24}$$

where the sum is over the permutation group S_N. On the other hand, the general solution of the Schrödinger equation (18) can be written as

$$\Psi(\{x, t\}) = \int \mathrm{d}^N x' w_N(\{x'\}) \mathcal{G}(\{x\}, t|\{x'\}, 0) \Psi(\{x', 0\}), \tag{25}$$

where the N-fermion Green's function $\mathcal{G}(\{x\}, t|\{x'\}, 0)$ is defined by

$$\left(\frac{\partial}{\partial t} + \mathcal{H}\right) \mathcal{G}(\{x\}, t|\{x'\}, 0) = 0, \tag{26}$$

with the anti-symmetric initial condition

$$\begin{aligned} \mathcal{G}(\{x\}, 0|\{x'\}, 0) &= \frac{1}{N! w_N(\{x\})} \sum_{P \in S_N} (-1)^P \prod_{i=1}^{N} \delta(x_i - x'_{P(i)}) \\ &= \frac{1}{N! w_N(\{x\})} \det[\delta(x_i - x'_j)]_{i,j=1,\ldots,N} \\ &= \frac{1}{N!} \det\left[\frac{\delta(x_i - x'_j)}{w(x_i)}\right]_{i,j=1,\ldots,N}. \end{aligned} \tag{27}$$

By comparing (22) and (25) and using the transformation (17) we obtain the following simple relation between the transition probability and Green's function

$$\mathcal{P}(\{x\}, t|\{x'\}, 0) = \mathrm{e}^{-E_0 t} w_N(\{x\}) \left(\frac{J_\beta(\{x\})}{J_\beta(\{x'\})}\right)^{1/2} \mathrm{sign}\left(\frac{\Delta_N(\{x\})}{\Delta_N(\{x'\})}\right) \mathcal{G}(\{x\}, t|\{x'\}, 0). \tag{28}$$

The problem is thus completely solved if we can obtain the N-fermions Green's function. This can be done exactly in the particular case $\beta = 2$, as we demonstrate in section 3. Other assignments to particle's statistics have also been used in the literature. For example, it is possible to treat Calogero–Sutherland models with rational couplings, $\beta = p/q$, by interpreting it as a description of a gas of free anyons [32].

3. Correlation functions of determinantal processes

For systems with unitary symmetry ($\beta = 2$) the two-body interaction term in the Calogero–Sutherland Hamiltonian (20) vanishes reducing the system to a free fermion gas and $\mathcal{H}$ to a sum of single-particle Hamiltonians $\mathcal{H} = \sum_{i=1}^{N} H_{x_i}$, where

$$H_x = -\frac{1}{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(w(x)s(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) = -s(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - r(x)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{29}$$

The one-particle Hamiltonian satisfies the eigenvalue equation

$$H_x\varphi_n(x) = -\epsilon_n\varphi_n, \tag{30}$$

with eigenenergies $\epsilon_n = s_2 n^2 + (r_1 - s_2)n$. The normalized wavefunctions $\varphi_n(x)$ form a orthogonal and complete set, so that

$$\int_a^b \mathrm{d}x\, w(x)\varphi_n(x)\varphi_m(x) = \delta_{n,m} \qquad \text{and} \qquad \sum_{n=0}^{\infty}\varphi_n(x)\varphi_n(y) = \frac{\delta(x-y)}{w(x)}. \tag{31}$$

The single-particle Green's function $g(x,t|y,t')$ defined by the initial value problem

$$\left(\frac{\partial}{\partial t} + H_x\right)g(x,t|y,t') = 0; \qquad g(x,t|y,t) = \frac{\delta(x-y)}{w(x)}, \tag{32}$$

has the following spectral representation

$$g(x,t|y,t') = \sum_{n=0}^{\infty}\varphi_n(x)\varphi_n(y)\,\mathrm{e}^{\epsilon_n(t-t')}. \tag{33}$$

Form (27) and (32), the N-particle Green's function $\mathcal{G}(\{x\},t|\{x'\},0)$ becomes a Slater determinant of a matrix with entries given by the single-particle Green's function $g(x_i,t|x'_j,0)$

$$\mathcal{G}(\{x\},t|\{x'\},0) = \frac{1}{N!}\det[g(x_i,t|x'_j,0)]_{i,j=1,\ldots,N}. \tag{34}$$

Thus we obtain a factorized representation for the transition probability

$$\mathcal{P}(\{x\},t|\{x'\},0) = \frac{\mathrm{e}^{-E_0 t}}{N!}\frac{\Delta_N(\{x\})}{\Delta_N(\{x'\})}w_N(\{x\})\det[g(x_i,t|x'_j,0)]_{i,j} \tag{35}$$

where

$$E_0 = \sum_{n=0}^{\infty}\epsilon_n = \frac{N(N-1)}{6}(2s_2(N-1)+3r_1). \tag{36}$$

In the literature of stochastic differential equations the determinantal structure of Green's function is obtained by means of the Karlin–McGregor formula and the final structure (35), resulting from the Sutherland's map, is known as the h-transform. This term has origin in the harmonic function $h(x)$ which is identical to our Vandermond determinant $\Delta(\{x\})$. We emphasize that, unlike the derivation using stochastic differential equation, (35) is not restricted to Hermitian matrix-valued processes. It can also be applied for solving the DMPK equation, which is a Fokker–Planck equation for a pseudo-unitay matrix-valued process, as we shall see in section 7.

4. A general method using biorthogonal functions

In this section we obtain an exact expression for the n-point correlation function (1) of orthogonal polynomial Brownian-motion ensembles with the delta-function initial condition. We avoid to face the BBGKY-like hierarchy (5) by using an extension of the biorthogonal functions method. This method was introduced by Muttalib [33] and used by Frahm [3] to obtain the correlation function of the DMPK equation, which is a Fokker–Planck equation describing the evolution, with sample length, of the transmission eigenvalue distribution in a quantum wire [20].

The main idea of the method is to try and write the transition probability (35) in a determinantal form

$$\mathcal{P}(\{x\}, t|\{x'\}, 0) = C_N \det[K(x_i, x_j; t)]_{i,j=1,\ldots,N}, \tag{37}$$

with a kernel $K(x, y; t)$ satisfying the properties

$$\int_a^b \mathrm{d}x\, K(x, x; t) = N, \tag{38}$$

$$\int_a^b \mathrm{d}z\, K(x, z; t) K(z, y; t) = K(x, y; t). \tag{39}$$

If such kernel exists then, according to a well known theorem of RMT [19], the normalization constant reads $C_N = 1/N!$ and the n-point correlation function can be written as

$$\rho_n(x_1, \ldots, x_n; t) = \det[K(x_i, x_j; t)]_{i,j=1,\ldots,n}. \tag{40}$$

In the next subsection we construct the kernel by generalizing the method of biorthogonal functions.

4.1. Kernel construction

Our starting point is the form (35) of the transition probability. Writing $\Delta_N(\{x\})$ as a Vandermonde determinant

$$\Delta_N(\{x\}) = (-1)^{N(N-1)/2} \det\left(x_i^{j-1}\right)_{i,j=1,\ldots,N}, \tag{41}$$

we can write

$$\mathcal{P}(\{x\}, t|\{x'\}, 0) = \frac{w_N(\{x\})}{N!} \frac{\det[\phi_{j-1}(x_k, t)]_{j,k} \det[\chi_{j-1}(x_k, t)]_{j,k}}{\det[\phi_{j-1}(x'_k, 0)]_{j,k}}, \tag{42}$$

where we defined the functions

$$\phi_n(x, t) = x^n \mathrm{e}^{-\epsilon_n t}, \qquad n = 0, \ldots, N-1, \tag{43}$$

$$\chi_m(x, t) = g(x, t|x'_{m+1}, 0), \qquad m = 0, \ldots, N-1. \tag{44}$$

The factor $\mathrm{e}^{-E_0 t} = \prod_n \mathrm{e}^{-\epsilon_n t}$ was incorporated in the matrix $\phi_{i-1}(x_j, t)$. Moreover, using the identity

$$\phi_{j-1}(x'_k, 0) = \int_a^b \mathrm{d}x\, w(x) \phi_{j-1}(x) \chi_{k-1}(x, 0) \tag{45}$$

we can write the transition probability as

$$\mathcal{P}(\{x\}, t|\{x'\}, 0) = \frac{w_N(\{x\})}{N!} \frac{\det[\phi_{j-1}(x_k, t)]_{j,k} \det[\chi_{j-1}(x_k, t)]_{j,k}}{\det[\int_a^b \mathrm{d}x\, w(x) \phi_{j-1}(x, 0) \chi_{k-1}(x, 0)]_{j,k}}. \tag{46}$$

We remark that this decomposition is not unique. We can perform elementary operations on the rows of the matrices without changing the determinant. With this freedom we can choose the new set of functions

$$\phi_n(x,t) \to \psi_n(x,t) = \int_a^b \mathrm{d}y\, w(y) L_{n+1}(y) g(y,0|x,t), \tag{47}$$

$n = 0, \ldots, N-1$, where $g(x,t|y,t')$ is the single particle Green's function and $L_n(x)$ is the Lagrange interpolation polynomial defined by

$$L_{n+1}(x) = \prod_{\substack{l=0 \\ l\neq n}}^{N-1} \frac{x - x'_{l+1}}{x'_{n+1} - x'_{l+1}}, \qquad n = 0, \ldots, N-1. \tag{48}$$

As shown in appendix A, the choice (47) ensures that the functions $\{\psi_n(x,t)\}$ and $\{\chi_m(x,t)\}$ form a biorthogonal set, i.e.

$$\int_a^b \mathrm{d}x\, w(x) \psi_n(x,t) \chi_m(x,t) = \delta_{n,m}, \tag{49}$$

therefore the denominator reduces to the determinant of the identity matrix $\det[\delta_{i,j}] = 1$. In this way we can write the transition probability in terms of the product of two determinants

$$\mathcal{P}(\{x\},t|\{x'\},0) = C_N \det[\psi_{j-1}(x_k,t) w(x_k)]_{j,k} \det[\chi_{j-1}(x_k,t)]_{k,j}. \tag{50}$$

Where the factor $w_N(\{x\}) = \prod_n w(x_n)$ was incorporated to the first matrix. As the product of two determinants is equal to the determinant of the product of the corresponding matrix and the determinant is invariant under matrix transposition, we can write the transition probability in the form (37) with kernel given by

$$K(x,y;t) = w(x) \sum_{n=0}^{N-1} \chi_n(x,t) \psi_n(y,t). \tag{51}$$

The biorthogonality condition (49) ensures that this kernel satisfies conditions (38) and (39), which allow us to write the n-point correlation function in the form (40).

Using the definitions of the functions $\chi_n(x,t)$ and $\psi_n(y,t)$ we can write the kernel in terms of the one-particle Green's function, so that

$$K(x,y;t) = w(x) \int_a^b \mathrm{d}\xi\, w(\xi) g(\xi,0|y,t) \sum_{n=1}^{N} L_n(\xi) g(x,t|x'_n,0), \tag{52}$$

which, with the aid of Cauchy integral formula, can be written as

$$K(x,y;t) = w(x) \int_a^b \mathrm{d}\xi\, w(\xi) g(\xi,0|y,t) \sum_{n=1}^{N} \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{L_n(\xi) g(x,t|z,0)}{z - x'_n}, \tag{53}$$

where the z-integral encircles the points x'_n in the complex plane. The dependence of the kernel on the Lagrange interpolation polynomial makes explicit calculation rather daunting. One of the features of our method is the elimination of this dependence using the identity

$$\Omega(\xi,z) \equiv \prod_{l=1}^{N} \frac{\xi - x'_l}{z - x'_l} = 1 - (z - \xi) \sum_{n=1}^{N} \frac{L_n(\xi)}{z - x'_n}, \tag{54}$$

which enables us to rewrite the kernel in terms of two coupled integrals involving the one-particle Green's function

$$K(x,y;t) = w(x) \int_a^b \mathrm{d}\xi\, w(\xi) \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{g(x,t|z,0) g(\xi,0|y,t)}{z - \xi} (1 - \Omega(\xi,z)). \tag{55}$$

This is our main result. In particular, for the degenerate initial condition, i.e., $x'_n = x_0$, $\forall n$, we can use the identity

$$\Omega(\xi, z) = \left(\frac{\xi - x_0}{z - x_0}\right)^N = 1 - (z - \xi)\sum_{l=0}^{N-1}\frac{(\xi - x_0)^l}{(z - x_0)^{l+1}}, \tag{56}$$

to obtain a factorization of the integrals of (55) and write the kernel as

$$K(x, y; t) = w(x)\sum_{l=0}^{N-1} I_l(x, t)J_l(y, t), \tag{57}$$

where

$$I_l(x, t) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}}\frac{g(x, t|z, 0)}{(z - x_0)^{l+1}} \tag{58}$$

$$J_l(y, t) = \int_a^b \mathrm{d}\xi\, w(\xi)(\xi - x_0)^l g(\xi, 0|y, t). \tag{59}$$

4.2. Sum representation

If the explicit calculation of the integrals (58) and (59) is not possible, we can use Green's function spectral decomposition (33) directly on the definitions of $\psi_n(x, t)$ and $\chi_n(x, t)$, yielding

$$\psi_n(x, t) = \sum_{m=0}^{N-1}\varphi_m(x)\,\mathrm{e}^{-\epsilon_m t}\int_a^b \mathrm{d}\xi\, w(\xi)L_{n+1}(\xi)\varphi_m(\xi), \tag{60}$$

$$\chi_n(x, t) = \sum_{m=0}^{\infty}\varphi_m(x)\varphi_m(x'_{n+1})\,\mathrm{e}^{\epsilon_m t}. \tag{61}$$

The first sum became finite because the integral $\int_a^b \mathrm{d}\xi\, w(\xi)L_{n+1}(\xi)\varphi_m(\xi)$ is zero for $m \geqslant N$, since $L_{n+1}(\xi)$ is a polynomial of degree $N - 1$. Substituting these expressions into the kernel (51) we find

$$K(x, y; t) = w(x)\sum_{m=0}^{N-1}\sum_{l=0}^{\infty}\varphi_l(x)\varphi_m(y)\,\mathrm{e}^{(\epsilon_l - \epsilon_m)t}\sum_{n=1}^{N}\int_a^b \mathrm{d}\xi\, w(\xi)\varphi_m(\xi)L_n(\xi)\varphi(x'_n). \tag{62}$$

Using the following property of the Lagrange interpolation polynomial

$$\sum_{n=1}^{N} L_n(\xi)\varphi_l(x'_n) = \begin{cases}\varphi_l(\xi), & l = 0, 1, \ldots, N-1,\\ \displaystyle\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}}\frac{\varphi_l(z)}{z - \xi}[1 - \Omega(\xi, z)], & l = N, N+1, N+2, \ldots\end{cases} \tag{63}$$

and with the orthogonality conditions of the polynomials $\varphi_n(x)$ we arrive at the final representation of the kernel

$$K(x, y; t) = w(x)\left(\sum_{l=0}^{N-1}\varphi_l(x)\varphi_l(y) + \sum_{m=0}^{N-1}\sum_{l=N}^{\infty} r_{lm}\varphi_l(x)\varphi_m(y)\,\mathrm{e}^{(\epsilon_l - \epsilon_m)t}\right), \tag{64}$$

where

$$r_{lm} = \int_a^b \mathrm{d}\xi\, w(\xi)\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}}\frac{\varphi_m(\xi)\varphi_l(z)}{z - \xi}[1 - \Omega(\xi, z)]. \tag{65}$$

In the particular case of degenerate initial conditions, the integrals factorize and we can write

$$r_{lm} = \sum_{n=m}^{N-1} a_{ln} b_{mn}, \tag{66}$$

where

$$a_{ln} = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{\varphi_l(z)}{(z - x_0)^{n+1}}, \tag{67}$$

$$b_{mn} = \int_a^b \mathrm{d}\xi \, w(\xi)(\xi - x_0)^n \varphi_m(\xi). \tag{68}$$

In the next section we give applications of the above formulae for Hermite, Laguerre and Jacobi ensembles with degenerate initial conditions. We remark that previous applications of the method of biorthogonal functions suffered from effects related to the non-uniqueness of the decomposition (46) and led to rather inconvenient expressions for the kernel, which involved explicitly the Lagrange interpolation polynomial. A systematic elimination of this difficulty is the main advantage of our approach.

5. Classical ensembles with degenerate initial condition

5.1. Hermite ensemble

From table 1 we see that the Hermite ensemble corresponds to the choice $a = -\infty, b = \infty, s(x) = 1$ and $w(x) = \mathrm{e}^{-x^2}$. The single-particle Hamiltonian is given by

$$H_x = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x}, \tag{69}$$

whose eigenfunctions are normalized Hermite polynomials

$$\varphi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) \tag{70}$$

with eigenenergies $\epsilon_n = -2n$. Thus, the single-particle Green's function reads

$$g(x, t|y, t') = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x) H_n(y) \, \mathrm{e}^{-2n(t-t')} \tag{71}$$

$$= \frac{\mathrm{e}^{x^2}}{\sqrt{\pi}\alpha_1(\tau)} \exp\left[-\frac{(x - \alpha_0(\tau)y)^2}{\alpha_1^2(\tau)}\right], \tag{72}$$

where $\alpha_0(t) = \mathrm{e}^{-2t}, \alpha_1(t) = \sqrt{1 - \mathrm{e}^{-4t}}$ and $\tau = t - t'$. We have thus two options for representing Green's function. In the following we choose the representation (72). In this case the kernel is given by (57) with coefficients given by the simple integrals:

$$I_l(x, t) = \frac{\mathrm{e}^{x^2}}{\sqrt{\pi}\alpha_1(t)} \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{\mathrm{e}^{-(x - \alpha_0(t)z)^2/\alpha_1^2(t)}}{(z - x_0)^{l+1}} \tag{73}$$

$$J_l(y, t) = \frac{1}{\sqrt{\pi}\alpha_1(-t)} \int_{-\infty}^{\infty} \mathrm{d}\xi (\xi - x_0)^l \, \mathrm{e}^{-(\xi - \alpha_0(-t)y)^2/\alpha_1^2(-t)}. \tag{74}$$

The first integral is solved by performing the transformation $z \to x_0 + \frac{\alpha_1(t)}{\alpha_0(t)} z$ and by using the following integral representation of the Hermite polynomial

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{l!}{z^{l+1}} \mathrm{e}^{-(z-x)^2} = \mathrm{e}^{-x^2} H_l(x), \tag{75}$$

yielding

$$I_l(x,t) = \frac{\mathrm{e}^{x^2}}{l!\sqrt{\pi\alpha_1^2(t)}} \left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^l \exp\left(-\frac{(x-\alpha_0(t)x_0)^2}{\alpha_1^2(t)}\right) H_l\left(\frac{x-\alpha_0(t)x_0}{\alpha_1(t)}\right). \tag{76}$$

The second integral is solved in a similar way. First, we apply the scale transformation $\xi \to x_0 + \alpha_1(-t)\xi$, then we eliminate the negative argument in α_0 and α_1 using the identities

$$\alpha_0(-t) = 1/\alpha_0(t) \tag{77}$$

$$\alpha_1(-t) = \mathrm{i}\alpha_1(t)/\alpha_0(t), \tag{78}$$

where $\mathrm{i} = \sqrt{-1}$. Finally we use the integral representation

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\sqrt{\pi}} (2\mathrm{i}\xi)^l \mathrm{e}^{-(\xi+ix)^2} = H_l(x) \tag{79}$$

to get the result

$$J_l(y,t) = \left(\frac{\alpha_1(t)}{2\alpha_0(t)}\right)^l H_l\left(\frac{y-\alpha_0(t)x_0}{\alpha_1(t)}\right). \tag{80}$$

Substituting (76) and (80) into (57) we obtain

$$K(x,y;t) = \frac{\mathrm{e}^{-(x-\alpha_0(t)x_0)^2/\alpha_1^2(t)}}{\sqrt{\pi\alpha_1^2(t)}} \sum_{l=0}^{N-1} \frac{1}{2^l l!} H_l\left(\frac{x-\alpha_0(t)x_0}{\alpha_1(t)}\right) H_l\left(\frac{y-\alpha_0(t)x_0}{\alpha_1(t)}\right), \tag{81}$$

which agrees with the result obtained from the Baker–Forrester's representation (A.17). The sum over the Hermite polynomials can be done by using the Christofell-Darboux formula [31] and we finally obtain

$$K(x,y;t) = \frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}2^N(N-1)!} \frac{H_N(\xi)H_{N-1}(\eta) - H_{N-1}(\xi)H_N(\eta)}{x-y}, \tag{82}$$

where $\xi = (x-\alpha_0(t)x_0)/\alpha_1(t)$ and $\eta = (y-\alpha_0(t)x_0)/\alpha_1(t)$.

5.2. Laguerre ensemble

From table 1 we have $a = 0, b = \infty, s(x) = x$ and the weight function is $w(x) = x^\nu \mathrm{e}^{-x}$, with $\nu > -1$. The Hamiltonian reads

$$H_x = -x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (1+\nu-x)\frac{\mathrm{d}}{\mathrm{d}x}, \tag{83}$$

whose wavefunctions are normalized Laguerre polynomials

$$\varphi_n(x) = \left(\frac{n!}{\Gamma(n+\nu+1)}\right)^{1/2} L_n^\nu(x) \tag{84}$$

with eigenenergies $\epsilon_n = -n$. Thus the one-particle Green's function reads

$$g(x,t|y,t') = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\nu+1)} L_n^\nu(x) L_n^\nu(y) \mathrm{e}^{-n(t-t')}. \tag{85}$$

As in the Hermite case, this summation can be performed exactly yielding

$$g(x,t|y,t') = \frac{\alpha^{1+\nu}(\tau)}{\Gamma(1+\nu)}\exp\left[(1-\alpha(\tau))(x+y)\right]{}_0F_1\big(1+\nu;\alpha(\tau)(\alpha(\tau)-1)xy\big), \tag{86}$$

where $\alpha(\tau) = (1-\mathrm{e}^{-\tau})^{-1}$ and $\tau = t-t'$. As in the Hermite case we choose to work with the representation (86) instead of (85). In the particular case of the degenerate initial condition with $x_0 = 0$, the kernel is given by the summation (57) with coefficients given by

$$I_l(x,t) = \frac{\alpha^{1+\nu}(t)}{\Gamma(1+\nu)}\,\mathrm{e}^{(1-\alpha(t))x}\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{(1-\alpha(t))z}\,{}_0F_1(1+\nu;\alpha(t)(\alpha(t)-1)xz)}{z^{l+1}}$$

$$J_l(y,t) = \frac{\alpha^{1+\nu}(-t)}{\Gamma(1+\nu)}\,\mathrm{e}^{(1-\alpha(-t))y}\int_0^\infty \mathrm{d}\xi\,\xi^{\nu+l}\,\mathrm{e}^{-\alpha(-t)\xi}\,{}_0F_1(1+\nu;\alpha(-t)(\alpha(-t)-1)\xi y).$$

To solve the first integral we perform the scale transformation $z \to z/(1-\alpha(t))$. In what follows we use the following integral representation of the Laguerre polynomial

$$\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{e^z\,{}_0F_1(1+\nu;-xz)}{z^{l+1}} = \frac{\Gamma(\nu+1)}{\Gamma(l+\nu+1)}L_l^\nu(x), \tag{87}$$

which yields

$$I_l(x,t) = \frac{\alpha^{1+\nu}(t)(1-\alpha(t))^l}{\Gamma(l+\nu+1)}\,\mathrm{e}^{(1-\alpha(t))x}L_l^\nu(\alpha(t)x). \tag{88}$$

The second integral is solved by implementing the scaling transformation $\xi \to \xi/\alpha(-t)$ followed by the substitution $\alpha(-t) = 1-\alpha(t)$ and by making use of the integral

$$\int_0^\infty \mathrm{d}\xi\,\mathrm{e}^{-\xi}\xi^{\nu+l}\,{}_0F_1(1+\nu;-\xi x) = l!\Gamma(\nu+1)\,\mathrm{e}^{-x}L_l^\nu(x), \tag{89}$$

yielding

$$J_l(y,t) = \frac{l!}{(1-\alpha(t))^l}L_l^\nu(\alpha(t)y). \tag{90}$$

Substituting (88) and (90) into (57) we obtain

$$K(x,y;t) = \alpha(t)(\alpha(t)x)^\nu\,\mathrm{e}^{-\alpha(t)x}\sum_{l=0}^{N-1}\frac{l!}{\Gamma(l+\nu+1)}L_l^\nu(\alpha(t)x)L_l^\nu(\alpha(t)y), \tag{91}$$

which agree with (A.26), obtained from the Baker–Forester representation. The sum can be done by using of the Christoffel–Darboux identity, yielding

$$K(x,y;t) = \frac{N!}{\Gamma(N+\nu)}(\alpha(t)x)^\nu\,\mathrm{e}^{-\alpha(t)x}\frac{L_{N-1}^\nu(\alpha(t)x)L_N^\nu(\alpha(t)y) - L_N^\nu(\alpha(t)x)L_{N-1}^\nu(\alpha(t)y)}{x-y}. \tag{92}$$

5.3. Jacobi ensemble

From table 1 we see that the Jacobi ensemble corresponds to the choice $a=-1$, $b=1$, $s(x) = 1-x^2$, $w(x) = (1-x)^\nu(1+x)^\mu$ and $r(x) = \mu-\nu-(2+\mu+\nu)x$. The single-particle Hamiltonian is given by

$$H_x = (x^2-1)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\nu-\mu+(\nu+\mu+2)x)\frac{\mathrm{d}}{\mathrm{d}x}, \tag{93}$$

whose eigenfunctions are normalized Jacobi polynomials

$$\varphi_n(x) = \frac{1}{\sqrt{h_n}}P_n^{(\nu,\mu)}(x), \tag{94}$$

where

$$h_n = \frac{2^{\nu+\mu+1}\Gamma(n+\nu+1)\Gamma(n+\mu+1)}{n!(2n+\nu+\mu+1)\Gamma(n+\nu+\mu+1)}, \tag{95}$$

with eigenenergies $\epsilon_n = -n(n+\nu+\mu+1)$. The one-particle Green's function reads

$$g(x,t|y,t') = \sum_{n=0}^{\infty} \frac{1}{h_n} P_n^{(\nu,\mu)}(x) P_n^{(\nu,\mu)}(y)\, \mathrm{e}^{\epsilon_n(t-t')}. \tag{96}$$

Unlike Hermite and Laguerre cases, there is not a summation formula for Jacobi polynomials and Green's function (96) can not be written in a closed form. In this case, the kernel can be written in the representation (64), which, for the degenerate initial condition, has the form

$$K(x,y;t) = (1-x)^{\nu}(1+x)^{\mu}\left(\sum_{l=0}^{N-1} \frac{1}{h_n} P_n^{(\nu,\mu)}(x) P_n^{(\nu,\mu)}(y) + \sum_{n=0}^{N-1}\sum_{m=N}^{\infty}\sum_{l=n}^{N-1} \frac{a_{nl}b_{ml}}{h_n h_m} P_n^{(\nu,\mu)}(x) P_m^{(\nu,\mu)}(y)\, \mathrm{e}^{(\epsilon_n-\epsilon_m)t}\right). \tag{97}$$

The coefficients, in the particular case of the degenerate initial condition with $x_0 = 1$, are given by

$$a_{nl} = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{P_n^{(\nu,\mu)}(z)}{(z-1)^{l+1}} = \frac{\Gamma(\nu+\mu+n+l+1)\Gamma(\nu+n+1)}{2^l l!(n-l)!\Gamma(\nu+\mu+n+1)\Gamma(\nu+l+1)}, \tag{98}$$

and

$$\begin{aligned} b_{nl} &= (-1)^l \int_{-1}^{1} \mathrm{d}\xi (1-\xi)^{\nu+l}(1+\xi)^{\mu} P_n^{(\nu,\mu)}(\xi) \\ &= (-1)^{n+l} 2^{\mu+\nu+l+1} \frac{l!\Gamma(\nu+l+1)\Gamma(\mu+n+1)}{n!(l-n)!\Gamma(\nu+\mu+l+n+2)}. \end{aligned} \tag{99}$$

Substituting (98) and (99) into (97) we get the final expression

$$K(x,y;t) = (1-x)^{\nu}(1+x)^{\mu}\left(\sum_{n=0}^{N-1} \frac{1}{h_n} P_n^{(\nu,\mu)}(x) P_n^{(\nu,\mu)}(y) + \sum_{n=0}^{N-1}\sum_{m=N}^{\infty} \frac{\Gamma(\nu+n+1)\Gamma(\mu+m+1)}{h_n h_m m!\Gamma(\nu+\mu+n+1)} D_{nm} P_n^{(\nu,\mu)}(x) P_m^{(\nu,\mu)}(y) \mathrm{e}^{(\epsilon_n-\epsilon_m)t}\right), \tag{100}$$

where the coefficient D_{nm} is obtained by summing over l

$$D_{nm} = 2^{\nu+\mu+1} \sum_{l=m}^{N-1} \frac{(-1)^{l+m}\Gamma(\nu+\mu+n+l+1)}{(l-m)!(n-l)!\Gamma(\nu+\mu+m+l+2)}, \tag{101}$$

yielding

$$D_{nm} = \frac{2^{\nu+\mu+1}(-1)^{N+m+1}(N-m)\Gamma(N+n+\nu+\mu+1)}{(n-N)!(N-m)!(n+m+\nu+\mu+1)(n-m)\Gamma(N+m+\nu+\mu+1)}. \tag{102}$$

The particular case of Legendre ensemble ($\nu = 0 = \mu$) was studied in [3] with a more involved method.

6. Connection with the Harish–Chandra–Itzykson–Zuber integral

There is an important connection between the determinantal processes discussed in the previous section and the famous Harish–Chandra–Itzykson–Zuber integral which we now discuss. Since it was introduced in the context of random matrix theory, the Itzykson-Zuber group integral, later recognized as a particular case of the result of Harish–Chandra, had been used in several areas of physics and mathematics (see [25] and references therein). The HCIZ integral appears naturally in random matrix theory as the transition probability of the Brownian-motion ensembles. As a concrete example, we cite the famous model introduced by Dyson [1] of an Hermitian matrix-valued Ornstein–Uhlenbeck process, which corresponds to the Hermite ensembles in our classification. This process is described by the following Fokker–Planck equation in the flat space of Hermitian matrices

$$\frac{\partial W}{\partial \tau} = \sum_i \left(-\frac{\partial}{\partial H_\mu} D_\mu^{(1)} + \frac{\partial^2}{\partial H_\mu^2} D_\mu^{(2)} \right) W, \tag{103}$$

where μ is a label for independent matrix elements, and the drift and diffusion coefficients given by $D_\mu^{(1)} = -2H_\mu$ and $D_\mu^{(2)} = \frac{1}{2}(1+\delta_{ij})$. If the initial distribution is $W(H,0) = W_0(H_0)$, the distribution in time τ is given by

$$W(H,\tau) = \int \mathrm{d}H_0 W(H,\tau|H_0) W_0(H_0), \tag{104}$$

where the volume element $\mathrm{d}H_0$ is the product of the differentials of the independent matrix elements of H_0. The transition probability satisfies the same Fokker–Planck equation, but with the delta initial condition: $W(H,\tau=0|H_0) = \delta(H-H_0)$. The transition amplitude of this Ornstein–Uhlenbeck process is known [30] to be

$$W(H,\tau|H_0) = C_{N,\tau} \exp\left(-\frac{\mathrm{Tr}(H-\alpha_0(\tau)H_0)^2}{\alpha_1^2(\tau)} \right), \qquad C_{N,\tau} = \frac{2^{N(N-1)/2}}{\left[\pi\alpha_1^2(\tau)\right]^{N^2/2}}, \tag{105}$$

where the $\alpha_0(\tau) = \mathrm{e}^{-2\tau}$ and $\alpha_1(\tau) = \sqrt{1-\mathrm{e}^{-4\tau}}$ were previously introduced in section 5.1. Note that, in the limit $\tau \to \infty$, the transition amplitude does not depend on H_0, and provided that $W_0(H_0)$ is normalized, the distribution $W(H,\tau)$ relaxes to its equilibrium value, which is a member from the Gaussian unitary ensemble (GUE). Exploring this property, we can study the crossover between different symmetry classes by choosing appropriately the initial distribution $W_0(H_0)$, for example we can choose H_0 to be a member from the Gaussian orthogonal ensemble (GOE). Using this idea the crossover GOE $\to$ GUE was thoroughly studied by Mehta and Pandey [21].

To describe the dynamics of the eigenvalues of H we proceed by decomposing the matrices $H = UXU^\dagger$ and $H_0 = U_0 X_0 U_0^\dagger$, where $X = \mathrm{diag}(x_1,\ldots,x_N)$ and U is an orthogonal, unitary or symplectic matrix for $\beta = 1$, 2 or 4, respectively. The differential probability satisfies the following equation [19]

$$W(H\tau)\mathrm{d}H = C_{N,\beta}|\Delta_N(X)|^\beta W(UXU^\dagger\tau)\,\mathrm{d}X\,\mathrm{d}U, \tag{106}$$

where $\mathrm{d}U$ is the normalized Haar's measure $\left(\int \mathrm{d}U = 1\right)$ of the appropriate group, $C_{N,\beta}$ is a normalization constant fixed by the resulting Selberg's integral and $\Delta_N(X)$ is the Vandermond determinant defined in (8). The reduced eigenvalue distribution is obtained by integrating over the angular part

$$P(X,\tau) = C_{N,2}|\Delta_N(X)|^2 \int \mathrm{d}U\; W(UXU^\dagger,\tau). \tag{107}$$

If the initial condition does not depend on the eigenvectors, $W_0(H_0 = U_0 X_0 U_0^\dagger) = W_0(X_0)$, we can write the eigenvalue distribution as

$$P(X,\tau) = C_{N,2} C_{N,\beta} |\Delta_N(X)|^2 \times \int \mathrm{d}U \int \mathrm{d}X_0 |\Delta_N(X_0)|^\beta W_0(X_0) \int \mathrm{d}U_0 W\big(UXU^\dagger, \tau \,\big|\, U_0 X_0 U_0^\dagger\big). \tag{108}$$

The angular dependence of the transition function can be worked out as follows. From the cyclic property of the trace in (104) we can write

$$W\big(UXU^\dagger, \tau \,\big|\, U_0 X_0 U_0^\dagger\big) = W\big(U_0^\dagger UXU^\dagger U_0, \tau \,\big|\, X_0\big), \tag{109}$$

and, since the measure remains invariant under the change of variables $U \to U_0 U$, the group integral reduces to

$$\Gamma(X,\tau|X_0) = \int \mathrm{d}U\, W(UXU^\dagger, \tau|X_0) \tag{110}$$

$$= C_{N,\tau} \exp\left(-\frac{\operatorname{Tr} X^2 + \alpha_0^2(\tau) \operatorname{Tr} X_0^2}{\alpha_1^2(\tau)}\right) I_\tau(X|X_0), \tag{111}$$

reducing the problem to the Itzykson–Zuber integral over the unitary group

$$I_\tau(X|X_0) = \int \mathrm{d}U \exp\left(\frac{2\alpha_0(\tau)}{\alpha_1^2(\tau)} \operatorname{Tr}(UXU^\dagger X_0)\right). \tag{112}$$

To sum up, the stochastic subprocess on the radial coordinates is described by

$$P(X,\tau) = \int \mathrm{d}X_0 \mathcal{P}(X,\tau|X_0) P_0(X_0), \tag{113}$$

where we defined the initial condition as

$$P_0(X_0) = C_{N,\beta} |\Delta_N(X_0)|^\beta W_0(X_0) \tag{114}$$

and the transition function is given by

$$\mathcal{P}(X,\tau|X_0) = C_{N,2} C_{N,\tau} |\Delta_N(X)|^2 \exp\left(-\frac{\operatorname{Tr} X^2 + \alpha_0^2(\tau) \operatorname{Tr} X_0^2}{\alpha_1^2(\tau)}\right) I_\tau(X|X_0), \tag{115}$$

thus establishing the connection between the Brownian-motion ensembles studied in this paper and the HCIZ integrals, whose solution depends on the solution of the corresponding Fokker–Planck equation.

The Fokker–Planck equation for the eigenvalues can be obtained by changing the Cartesian variables H in the Fokker–Planck equation (103) to their polar representation $H = UXU^\dagger$, which require the construction of the corresponding Laplace–Beltrami operator. Alternatively we may construct the matrix-valued stochastic subprocess associated with the eigenvalues of H. Either way, we end up with a Fokker–Planck equation with drift and diffusion coefficients given respectively by

$$D_i^{(1)} = -2x_i + 2 \sum_{j(\neq i)} \frac{1}{x_i - x_j} \qquad \text{and} \qquad D_i^{(2)} = 1. \tag{116}$$

As expected, these coefficients are exactly those of (13) evaluated at $\beta = 2$ (unitary symmetry) and $s(x) = 1$ and $r(x) = -2x$ (Hermite ensemble). In section 3, we showed that, in this case, the transition probability can be represented as

$$\mathcal{P}(\{x\},\tau|\{x_0\}) = \frac{\mathrm{e}^{-E_0\tau}}{N!} \frac{\Delta_N(\{x\})}{\Delta_N(\{x_0\})} w_N(\{x\}) \det[g(x_i,t|x_{0,j},0)]_{i,j=1,\ldots,N}, \tag{117}$$

where $E_0 = -N(N-1)$ and $w_N(\{x\}) = \prod_i \mathrm{e}^{-x_i^2} = \mathrm{e}^{-\operatorname{Tr} X^2}$. Inserting the single-particle Green's function (see representation (72))

$$g(x,t|y,0) = \frac{\mathrm{e}^{x^2}}{\sqrt{\pi\alpha_1^2(t)}} \exp\left(-\frac{(x-\alpha_0(\tau)y)^2}{\alpha_1^2(\tau)}\right) \tag{118}$$

in (117) and after some rearrangements we obtain

$$\mathcal{P}(X,t|X_0) = \frac{\mathrm{e}^{N(N-1)\tau}}{N!}\frac{\Delta_N(X)}{\Delta_N(X_0)}\,\mathrm{e}^{-(\operatorname{Tr} X^2+\alpha_0^2(\tau)\operatorname{Tr} X_0^2)/\alpha_1^2(\tau)}\det[\mathrm{e}^{\lambda_\tau x_i x_{0,j}}]_{i,j=1,\ldots,N}, \tag{119}$$

where we defined $\lambda_\tau = 2\alpha_0(\tau)/\alpha_1^2(\tau)$. Comparing (119) and (115) we recover the Itzykson–Zuber result

$$I_\tau(X,X_0) = \int \mathrm{d}U \exp[\lambda_\tau \operatorname{Tr}(UXU^\dagger X_0)] = \frac{\prod_{n=1}^{N-1} n!}{\lambda_\tau^{N(N-1)/2}}\frac{\det[\mathrm{e}^{\lambda_\tau x_i x_{0,j}}]}{\Delta_N(X)\Delta_N(X_0)}. \tag{120}$$

The crossover problems whose solutions are known, GOE $\to$ GUE [21], GSE $\to$ GUE [22], and the corresponding problems in circular ensembles [23], are all examples of free fermion dynamics. Their solutions are equivalent to solving the HCIZ integral in the unitary group ($\beta = 2$). The other cases corresponding to HCIZ integrals over orthogonal and symplectic groups correspond to the dynamics of interacting fermions and are still open. A deduction of the HCIZ formula is also available in the literature of stochastic differential equations [24]. For a recent discussion using the color-flavor transformation see [34].

7. Transfer matrix ensembles

In [11] we showed how to extend the methods developed in the context of polynomial ensembles to random transfer ensembles, unifying the calculational methods and providing a classification scheme alternative to Cartan's table of symmetric spaces. In this section we continue this procedure by adapting the generalized method of biorthogonal functions to obtain the n-point correlation functions for non-polynomial ensembles described by the DMPK equation, which is a Fokker–Planck equation describing the Brownian motion of the transmission eigenvalues (eigenvalues of $T^\dagger T$, where T is the transmission matrix) of a quantum wire, as its length is increased, see [20] for a review.

In a previous section we applied our method to solve crossover problems from a symmetrized delta distribution to the joint distribution of the classical orthogonal polynomial ensembles. Despite its mathematical interest, these problems do not have a direct physical application. However, a crossover with a degenerate, delta-like initial condition is the one naturally occurring in the DMPK equation. Although there is no equilibrium solution, we show in [11] that the transfer matrix ensembles can be treated on the same footing as polynomial ones if we introduce some special coordinates. In such coordinates, DMPK equations of all symmetry classes can be written in the standard form

$$\frac{\partial P}{\partial t} = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(J_\beta w_N s(x_i)\frac{\partial}{\partial x_i}\frac{P}{w_N J_\beta}\right), \tag{121}$$

where, as in section 2, each symmetry class is characterized by the functions $w(x)$, $s(x)$ and $r(x)$. Table 2 shows the classification of the DMPK symmetry classes.

In this case, the function $w(x)$ is not related to the weight function of classical orthogonal polynomials and condition (10) is not fulfilled, which allow non-polynomial solutions. The 'levels' $1 \leqslant x_i \leqslant \infty$ are related to the transmission eigenvalues, $0 \leqslant \tau_i \leqslant 1$, by $\tau_i = 2/(1+x_i)$, and N denotes the number of scattering channels.

Table 2. Classification of Brownian-motion ensembles associated with the DMPK equations.

Interval	$w(x)$	$s(x)$	$r(x)$	Ensemble
$[1,\infty)$	1	x^2-1	$2x$	Wigner–Dyson
$[1,\infty)$	$x^{[(1-N)\beta-2]/2}$	x^2	$[1-\beta(N-1)/2]x$	Chiral
$[1,\infty)$	$(x^2-1)^{(\alpha-1)/2}$	x^2-1	$(1+\alpha)x$	Bdg

The DMPK equation with the ballistic initial condition was first solved, for the unitary ensemble of the Wigner–Dyson class, in [4]. The equivalent solution in Chiral and BdG classes can be found in [5] and [6], respectively. By ballistic initial condition we mean that all transmission eigenvalues equal to unit, which in our coordinates is equivalent to solving (121) with the initial condition

$$P(\{x\},0)=\prod_{i=1}^{N}\delta(x_i-1). \tag{122}$$

The method of biorthogonal functions, developed in section 4 for polynomial ensembles with unitary symmetry ($\beta=2$), can be adapted to find the n-point correlation functions for DMPK problems. In this paper we address the problem for a generic interpolation class defined by

$$w(x)=(x^2-1)^{(\alpha-1)/2}, \qquad s(x)=x^2-1 \qquad \text{and} \qquad r(x)=(1+\alpha)x. \tag{123}$$

According to table 1 of [11], the case $\beta=2$ corresponds to the BdG classes C and D, associated with systems with time reversal (TR) invariance and identified by the parameter $\alpha=0$ or $\alpha=2$ according to the absence or presence of spin-rotation (SR) invariance, respectively. The case $\alpha=1$ is also of interest since it implies $w(x)=1$, which according to table 2 corresponds to the Wigner–Dyson (WD) class. Physically, the WD class with $\beta=2$ describes systems with broken TR symmetry. In the next subsection we calculate an explicit formula for the n-point correlation which is valid for arbitrary values of N and $\alpha>-1$. This expression was presented without derivation in [35].

7.1. The n-point correlation function

When written in normal coordinates the DMPK equation has the same mathematical structure as the Fokker–Planck equation defined for polynomial ensembles. As a result we may follow the same steps taken in section 3, which yields the following determinantal structure for the transition probability

$$\mathcal{P}(\{x\},t|\{x'\},0)=\frac{\mathrm{e}^{-E_0 t}}{N!}\frac{\Delta(\{x\})}{\Delta(\{x'\})}w_N(\{x\})\det[g(x_i,t|x'_j,0)]_{i,j=1,\ldots,N}, \tag{124}$$

where

$$E_0=\frac{N(N-1)}{6}(2N+3\alpha-1)=\sum_{n=0}^{N-1}\epsilon_n, \tag{125}$$

with $\epsilon_n=n(n+\alpha)$. To determine the one-particle Green's function that appear in the determinant we need to study the following Hamiltonian

$$H_x=-s(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}-r(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}=(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}-(1+\alpha)x\frac{\mathrm{d}}{\mathrm{d}x}. \tag{126}$$

Non-normalizable wavefunctions are characteristic of DMPK problems. In this case the one-particle Hamiltonian satisfies the eigenvalue equation $H_x\varphi_k(x) = \varepsilon_k\varphi_k(x)$, with continuum spectra

$$\varepsilon_k = k^2 + \alpha^2/4, \qquad k \geqslant 0. \tag{127}$$

The eigenfunctions are given in terms of the hypergeometric function

$$\varphi_k(x) = A_k\mathcal{F}_k^{(\alpha)}(x); \qquad \mathcal{F}_k^{(\alpha)}(x) = {}_2F_1\left(\frac{\alpha}{2} + ik, \frac{\alpha}{2} - ik; \frac{1+\alpha}{2}; \frac{1-x}{2}\right). \tag{128}$$

The constant A_k can be determined by the same method used in [36], yielding

$$A_k = \frac{2^{(1-\alpha)/2}|\Gamma(\alpha/2 + ik)|}{\Gamma(\alpha/2 + 1/2)|\Gamma(ik)|}. \tag{129}$$

Therefore, the one-particle wavefunction satisfies the orthogonality and completeness relations

$$\int_1^\infty \mathrm{d}x\, w(x)\varphi_k(x)\varphi_{k'}(x) = \delta(k - k'), \tag{130}$$

$$\int_0^\infty \mathrm{d}k\, \varphi_k(x)\varphi_k(x') = \delta(x - x'). \tag{131}$$

The one-particle Green's function has thus the following spectral decomposition

$$g(x, t|y, t') = \int_0^\infty \mathrm{d}k\, \mathrm{e}^{-\varepsilon_k(t-t')}\varphi_k(x)\varphi_k(y). \tag{132}$$

The continuum spectra make the approach to the DMPK equation somewhat different from that used in the polynomial ensemble. Furthermore, the same single-particle Hamiltonian satisfies another eigenvalue equation: $H_x\phi_n = -\epsilon_n\phi_n$, with a discrete spectrum

$$\epsilon_n = n(n + \alpha), \qquad n = 0, 1, 2, \ldots, \tag{133}$$

whose eigenfunctions are normalized Jacobi polynomials

$$\phi_n(x) = \frac{1}{\sqrt{h_n}}P_n^{(\nu,\nu)}(x), \qquad h_n = \frac{2^{2\nu+1}\big(\Gamma(n + \nu + 1)\big)^2}{n!(2n + 2\nu + 1)\Gamma(n + 2\nu + 1)}, \tag{134}$$

with parameter $\nu = (\alpha - 1)/2$. Rewriting the Slater determinant in terms of these polynomial eigenfunctions we find

$$\Delta(\{x\}) \propto \det[\phi_{i-1}(x_j)]_{i,j=1,\ldots,N}, \tag{135}$$

therefore we can write (124) as

$$P(\{x\}, t|\{x'\}, t) = \frac{w_N(\{x\})}{N!}\frac{\det[\phi_{i-1}(x_j, t)]\det[\chi_{j-1}(x_k, t)]}{\det[\phi_{i-1}(x'_j, 0)]}, \tag{136}$$

in which we have introduced the functions

$$\phi_n(x, t) = \mathrm{e}^{-\epsilon_n t}\phi_n(x), \qquad n = 0, 1, \ldots, N - 1 \tag{137}$$

$$\chi_m(x, t) = g(x, t|x'_{m+1}, 0), \qquad m = 0, 1, \ldots, N - 1, \tag{138}$$

which satisfy the following initial value problems

$$\left(\frac{\partial}{\partial t} - H\right)\phi_n(x, t) = 0, \qquad \phi_n(x, 0) = \phi_n(x) \tag{139}$$

$$\left(\frac{\partial}{\partial t}+H\right)\chi_m(x,t)=0, \qquad \chi_m(x,0)=\frac{\delta(x-x'_{m+1})}{w(x)}. \tag{140}$$

Following Frahm we define the overlap integral of the functions $\phi_n(x,t)$ and $\chi_m(x,t)$ by

$$A_{nm}(t)=\int_1^\infty \mathrm{d}x\; w(x)\phi_n(x,t)\chi_m(x,t). \tag{141}$$

It is convenient to introduce the two-time auxiliary function

$$\tilde{A}_{nm}(t,t')=\int_1^\infty \mathrm{d}x\; w(x)\phi_n(x,t)\chi_m(x,t'). \tag{142}$$

Note that $A_{nm}(t)=\tilde{A}_{nm}(t,t)$. Using equations (140) and (139) we find the following relation between the time derivatives of $\tilde{A}$

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{A}_{nm}(t,t') &= \int_1^\infty \mathrm{d}x\; w(x)\chi_m(x,t')H\phi_n(x,t)\\ &= \int_1^\infty \mathrm{d}x\; w(x)\phi_n(x,t)H\chi_m(x,t') = -\frac{\partial}{\partial t'}\tilde{A}_{nm}(t,t'),\end{aligned}$$

which allows us to conclude that $\tilde{A}$ depends only on the time difference, i.e, $\tilde{A}_{nm}(t,t')=\tilde{A}_{nm}(t-t')$. In particular, for equal times, the overlap function (141) does not depend on time and the integral can be evaluated for $t=0$, yielding

$$A_{nm}(t)=A_{nm}(0)=\phi_n(x'_{m+1}). \tag{143}$$

Therefore the functions $\{\phi_n\}$ and $\{\chi_n\}$ clearly do not form a biorthogonal set. In his pioneering paper Frahm adapted the method introduced by Muttalib to construct such a set. Here we proceed as in section 4. We make appropriate linear combination of the matrix elements in order to diagonalize the overlap function. As shown in appendix A, the desired transformation is

$$\phi_n(x,t)\to\psi_n(x,t)=\int_{-1}^1 \mathrm{d}y\; v(y)L_{n+1}(y)g_2(x,t|y,0), \qquad n=0,1,\ldots,N-1, \tag{144}$$

where

$$g_2(x,t|y,0)=\sum_{n=0}^{\infty}\mathrm{e}^{-\epsilon_n t}\phi_n(x)\phi_n(x) \tag{145}$$

is Green's function satisfying

$$\left(\frac{\partial}{\partial t}-H\right)g_2(x,t|x',t')=0; \qquad g_2(x,t|y,t)=\frac{\delta(x-y)}{v(x)}. \tag{146}$$

By definition $\psi_{i-1}(x'_j,0)=\delta_{ij}$ and the matrix in the denominator is mapped into the identity. By construction, the new set of functions $\{\psi_n\}$ and $\{\chi_n\}$ satisfy the biorthogonality condition

$$\int_1^\infty \mathrm{d}x\; w(x)\psi_n(x,t)\chi_m(x,t)=\delta_{n,m}, \tag{147}$$

therefore we can write the solution in the determinantal form

$$P(\{x\},t|\{x'\},0)=\frac{1}{N!}\det[K(x_i,x_j;t)]_{i,j=1,\ldots,N}, \tag{148}$$

with the kernel given by

$$K(x,y;t)=w(x)\sum_{n=0}^{N-1}\psi_n(x,t)\chi_n(y,t). \tag{149}$$

Using the definitions of the functions $\chi_n(x,t)$ and $\psi_n(x,t)$ and following the same steps as in section 4 we arrive at the DMPK version of (55) for the kernel

$$K(x,y;t) = w(x)\int_{-1}^{1} \mathrm{d}\xi\, v(\xi) \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1-\Omega(\xi,z)}{(z-\xi)} g_1(y,t|z,0) g_2(x,t|\xi,0). \tag{150}$$

This equation is valid for arbitrary delta-function initial conditions. For degenerate (ballistic) initial conditions, $x_n' = 1$, the integrals factorize and we can write

$$K(x,y;t) = w(x)\sum_{l=0}^{N-1} I_l(x,t) J_l(y,t), \tag{151}$$

where

$$I_l(x,t) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\epsilon_n t}}{h_n} a_{ln} P_n^{(\nu,\nu)}(x), \tag{152}$$

and

$$J_l(y,t) = \int_0^{\infty} \mathrm{d}k A_k^2\, \mathrm{e}^{-\varepsilon_k t} b_l(k) \mathcal{F}_k^{(\alpha)}(y). \tag{153}$$

The coefficients are given by

$$\begin{aligned} a_{ln} &= \int_{-1}^{1} \mathrm{d}x (1-x^2)^{\nu} (x-1)^l P_n^{(\nu,\nu)}(x) \\ &= \frac{(-1)^{n+l} 2^{l+\alpha} l! \Gamma(l+\alpha/2+1/2)\Gamma(n+\alpha/2+1/2)}{(l-n)! n! \Gamma(n+l+\alpha+1)} \theta(l-n) \end{aligned} \tag{154}$$

and

$$b_l(k) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{\mathcal{F}_k(z)}{(z-1)^{l+1}} = \frac{(-1)^l |\Gamma(\alpha/2+\mathrm{i}k+l)|^2 \Gamma(\alpha/2+1/2)}{2^l l! |\Gamma(\alpha/2+\mathrm{i}k)|^2 \Gamma(l+\alpha/2+1/2)}. \tag{155}$$

Using these coefficients explicitly in (151) we find

$$\begin{aligned} K(x,y;t) = w(x) \sum_{l=0}^{N-1}\sum_{n=0}^{\infty} \int_0^{\infty} dk A_k^2 \mathcal{F}_k^{(\alpha)}(z)\, \mathrm{e}^{-\varepsilon_{nk} t} P_n^{(\nu,\nu)}(x) \frac{(-1)^n (2n+\alpha)\Gamma(n+\alpha)}{\Gamma(n+\alpha/2+1/2)} \\ \times \frac{\Gamma(\alpha/2+1/2)|\Gamma(\alpha/2+\mathrm{i}k+l)|^2}{\Gamma(n+l+\alpha+1)|\Gamma(\alpha/2+\mathrm{i}k)|^2 (l-n)!} \theta(l-n), \end{aligned} \tag{156}$$

in which we defined

$$\varepsilon_{nk} = \epsilon_n + \varepsilon_k = n(n+\alpha) + k^2 + \alpha^2/4. \tag{157}$$

The Heaviside function implies the constraint $n \leqslant l$. Thus we can handle the summations according to the rule

$$\sum_{l=0}^{N-1}\sum_{n=0}^{\infty} A_{ln}\theta(l-n) \to \sum_{n=0}^{N-1}\sum_{l=n}^{N-1} A_{ln}. \tag{158}$$

Furthermore, the summation over l can be performed explicitly by

$$\sum_{l=n}^{N-1} \frac{|\Gamma(\alpha/2+\mathrm{i}k+l)|^2}{(l-n)!\Gamma(n+l+\alpha+1)} = \frac{C^{(\alpha)}(N,n,k)}{\varepsilon_{nk}}, \tag{159}$$

where we defined

$$C^{(\alpha)}(N,n,k) = \frac{|\Gamma(N+\alpha/2+\mathrm{i}k)|^2}{(N-n-1)!\Gamma(N+n+\alpha)}. \tag{160}$$

Therefore, we write the kernel as

$$K(x, y; t) = \frac{2^{1-\alpha}(x^2-1)^{(\alpha-1)/2}}{\Gamma(\alpha/2+1/2)} \sum_{n=0}^{N-1} (-1)^n \frac{(2n+\alpha)\Gamma(n+\alpha)}{\Gamma(n+\alpha/2+1/2)} P_n^{(\nu,\nu)}(x) \times \int_0^\infty \frac{\mathrm{d}k}{\pi} \frac{k \sinh(k\pi)}{\varepsilon_{kn}} C^{(\alpha)}(N, n, k)\, \mathrm{e}^{-\varepsilon_{kn} t} \mathcal{F}_k^{(\alpha)}(y) \tag{161}$$

which agrees with equation (3) of [35], where it was used to calculate the first three moments of the conductance. In particular, for $\alpha = 1$, the hypergeometric function can be written in terms of the conical function

$$\mathcal{F}_k^{(1)}(y) = {}_2F_1(1/2+ik, 1/2-ik; 1; (1-y)/2) = P_{-1/2+ik}(y), \tag{162}$$

and the kernel (161) reduces to the solution of the Wigner–Dyson class first obtained by Frahm [3]:

$$K(x, y; t) = \sum_{n=0}^{N-1} (-1)^n (2n+1) P_n(x) \int_0^\infty \frac{\mathrm{d}k}{\pi} \frac{k \sinh(k\pi)}{\varepsilon_{kn}} C^{(1)}(N, n, k)\, \mathrm{e}^{-\varepsilon_{kn} t} P_{-1/2+ik}(y). \tag{163}$$

This result was also derived in [11] using the method of integral transform.

8. Conclusions

We studied the n-point correlation function of Brownian-motion ensembles describing determinantal processes. Despite the complicated hierarchic relations satisfied by non-equilibrium correlation functions, we show that for delta-function initial conditions they have a factorized structure, similar to that found for equilibrium ensembles, where the n-point function can be written in terms of a two-point function, known as kernel. This factorization was obtained by solving the Fokker–Planck equation for the transition probability and using a generalization of the biorthogonal function method. Our method differs from that used in previous works by the systematic elimination of the Lagrange interpolation polynomial, which allows for an elegant integral representation of the kernel, exhibiting in a very transparent way its dependence on the one-particle Green's function, thus showing a hidden mathematical structure. Another advantage of our construction is the simplicity of the implementation of degenerate initial conditions, contrasting with the subtle method used for instance in [3]. As applications we obtained exact solutions for the Hermite, Laguerre and Jacobi Brownian-motion ensembles with the degenerate initial condition. Our results for Hermite and Laguerre agree with the results obtained from Baker–Forrester's representation, presented in [29], which was obtained by using Jack polynomial methods. We also adapted our method to be used for transfer matrix ensembles. In particular, we calculated the correlation functions for the BdG ensembles with time reversal symmetry. In this case the homogeneous initial conditions for the transmission eigenvalues allow us to write the kernel in terms of two uncoupled integrals.

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Appendix A. Baker–Forrester's representation

From our discussion in section 2.1 it is clear that one way to solve the Fokker–Planck equation (23) is to construct the spectral decomposition of Green's function of the Calogero–Sutherland Hamiltonian (20). This problem was solved by Baker–Forrester [29], by writing the eigenfunctions in terms of multivariate generalizations of the classical polynomials. In this section we outline Baker–Forrester's solution in our notation for future reference.

The ground state wavefunction of (20) is $\Psi_0 = J_\beta^{1/2}$ with eigenvalue $-E_0$. The excited states are obtained by multiplying a symmetric function to the ground state wavefunction, $\Psi = \Psi_0\Phi$. In this way, Ψ and Φ_0 have the same behavior under exchange of particles. The action of the Hamiltonian operator (20) on the wavefunction Ψ is given by

$$\mathcal{H}\Psi = -E_0\Psi - \Psi_0\mathcal{L}_{\mathrm{FP}}^{\dagger}\Phi, \tag{A.1}$$

where the adjoint Fokker–Planck operator reads [11]

$$\mathcal{L}_{\mathrm{FP}}^{\dagger} = -J_\beta^{-1/2}(\mathcal{H}+E_0)J_\beta^{1/2} = \sum_{i=1}^{N}\left(s(x_i)\frac{\partial^2}{\partial x_i^2} + r(x_i)\frac{\partial}{\partial x_i} + \beta\sum_{j(\neq i)}\frac{S(X_i)}{x_i - x_j}\frac{\partial}{\partial x_i}\right). \tag{A.2}$$

The excited state wavefunction Ψ is an eigenfunction of $\mathcal{H}$, with energy ε, if the symmetric function Φ satisfies the eigenvalue equation

$$\mathcal{L}_{\mathrm{FP}}^{\dagger}\Phi = -E\Phi. \tag{A.3}$$

The eigenvalues of $\mathcal{H}$ and $\mathcal{L}_{\mathrm{FP}}^{\dagger}$ are related by a simple translation $\varepsilon = E - E_0$. The eigenvalue equation (A.3) coincides with usual definition of the multivariate orthogonal polynomials [29, 37–39], which form a complete set of eigenfunction normalized as

$$\int \mathrm{d}x^N J_\beta(\{x\})w_N(\{x\})\Phi_\lambda(\{x\})\Phi_{\lambda'}(\{x\}) = \delta_{\lambda,\lambda'}. \tag{A.4}$$

The quantum numbers are partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$ represented by Young diagrams. The parts λ_j's of a partition λ are ordered as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_N$. The degree of a partition λ is defined by $|\lambda| = \sum_j \lambda_j$. With the knowledge of the eigenfunctions and eigenvalues of the Hamiltonian $\mathcal{H}$ we can write the spectral decomposition of Green's function

$$\begin{aligned} G(\{x\},t|\{x'\},t') &= \sum_\lambda \mathrm{e}^{-\varepsilon_\lambda(t-t')}\Psi_\lambda(\{x\})\Psi_\lambda(\{x'\}) \\ &= \mathrm{e}^{E_0 t}J_\beta^{1/2}(\{x\})J_\beta^{1/2}(\{x'\})\sum_\lambda \mathrm{e}^{-E_\lambda(t-t')}\Phi_\lambda(\{x\})\Phi_\lambda(\{x'\}). \end{aligned} \tag{A.5}$$

The transition probability is obtained from Green's function (A.5) through the relation (28),[2] yielding

$$P(\{x\},t|\{x'\},t') = J_\beta(\{x\})w_N(\{x\})\sum_\lambda \mathrm{e}^{-E_\lambda(t-t')}\Phi_\lambda(\{x\})\Phi_\lambda(\{x'\}). \tag{A.6}$$

Baker and Forrester analyzed in detail the Hermite and Laguerre ensembles. Here we translate their explicit solution to our notation. Further details can be found in the original paper.

[2] Since we are not specifying the statistics of the particles, the sign function appearing in (28) is not necessary.

A.1. Hermite ensemble

From table 1 the Hermite ensemble is characterized by $w(x) = \mathrm{e}^{-x^2}, s(x) = 1$ and $r(x) = -2x$. It is useful to introduce the notation $w_N(\{x\}) = \mathrm{e}^{-\mathrm{Tr}\, X^2}$, where $X = \mathrm{diag}(x_1, x_2, \ldots, x_N)$. The polynomial part of the eigenfunctions are the normalized Hermite multivariate polynomials

$$\Phi_\lambda(\{x\}) = \frac{1}{\sqrt{h_\lambda^{(2/\beta)}}} H_\lambda(X, 2/\beta), \tag{A.7}$$

where

$$h_\lambda^{(2/\beta)} = \frac{2^{|\lambda|}|\lambda|!}{C_N C_\lambda^{(2/\beta)}(1^N)}; \qquad C_N^{-1} = \frac{\pi^{N/2}}{2^{\beta N(N-1)/4}} \prod_{j=1}^{N} \frac{\Gamma(1+\beta j/2)}{\Gamma(1+\beta/2)}. \tag{A.8}$$

The generalized Hermite polynomials, also known in the literature as Hi-Jack polynomials, can be defined by the action of an exponential operator on the Jack polynomials

$$H_\lambda(X, 2/\beta) = \frac{2^{|\lambda|}}{C_\lambda^{(2/\beta)}(1^N)} \exp\left(-\frac{1}{4}\sum_i \frac{\partial^2}{\partial x_i^2} - \frac{\beta}{4}\sum_{i\neq i} \frac{1}{x_i - x_j}\frac{\partial}{\partial x_i}\right) C_\lambda^{(2/\beta)}(X). \tag{A.9}$$

The transition probability is obtained by substituting (A.7) into (A.6). Using a summation identity [29], it can be written in closed form

$$P(\{x\}, t|\{x_0\}, 0) = \frac{C_N}{(1-\mathrm{e}^{-4t})^{Nq/2}} J_\beta(\{x\}) \exp\left(-\frac{\mathrm{Tr}\, X^2 + \mathrm{e}^{-4t}\,\mathrm{Tr}\, {X_0}^2}{1-\mathrm{e}^{-4t}}\right) \times {}_0\mathcal{F}_0^{(2/\beta)}\left(\frac{2\,\mathrm{e}^{-2t}X}{\sqrt{1-\mathrm{e}^{-4t}}}, \frac{X_0}{\sqrt{1-\mathrm{e}^{-4t}}}\right), \tag{A.10}$$

where $q = 1 + \beta(N-1)/2$ and

$${}_0\mathcal{F}_0^{(\alpha)}(X, Y) = \sum_\lambda \frac{1}{|\lambda|!} \frac{C_\lambda^{(\alpha)}(X) C_\lambda^{(\alpha)}(Y)}{C_\lambda^{(\alpha)}(1^N)} \tag{A.11}$$

is the generalized hypergeometric function. The solution (A.10) is general and valid for all β values. This shows the power of Jack polynomials technique in contrast with the more restrictive group theory methods which can be applied only to $\beta = 1, 2$ and 4. Despite its importance, it is not obvious how to construct the n-point correlation functions (1) from this representation. For the degenerate initial condition, $X_0 = \mathrm{diag}(x_0, x_0, \ldots, x_0)$, however, equation (A.10) can be written in terms of elementary functions

$$\mathcal{P}(\{x\}, t|\{x_0\}, 0) = \frac{C_N J_\beta(\{x\})}{(1-\mathrm{e}^{-4t})^{Nq/2}} \exp\left(-\frac{1}{(1-\mathrm{e}^{-4t})}\sum_{i=1}^{N}(x_i - \mathrm{e}^{-2t}x_0)^2\right), \tag{A.12}$$

which allows the calculation of the correlation functions. For example, in the unitary class we have $\beta = 2, q = N$, and we can write (A.12) in the familiar form

$$\mathcal{P}(\{x\}, t|\{x_0\}, 0) = A_N(t)\Delta_N^2(X) w_N(X, t), \tag{A.13}$$

where we defined $A_N(t) = C_N[\alpha_1(t)]^{-N^2}$, $w_N(X, t) = \prod_i w(x_i, t)$, and the weight function

$$w(x, t) = \exp\left(-\frac{(x - \alpha_0(t)x_0)^2}{\alpha_1^2(t)}\right), \tag{A.14}$$

with $\alpha_0(t) = \mathrm{e}^{-2t}$ and $\alpha_1(t) = \sqrt{1-\mathrm{e}^{-4t}}$. The orthonormal polynomials with respect to the weight $w(x,t)$ are time-dependent Hermite polynomials

$$p_n(x,t) = \left(\frac{1}{2^n n!\sqrt{\pi\alpha_1^2(t)}}\right)^{1/2} H_n\left(\frac{x-\alpha_0(t)x_0}{\alpha_1(t)}\right). \tag{A.15}$$

Applying the usual orthogonal polynomials method [19], we can write (A.13) as a determinantal process

$$\mathcal{P}(\{x\},t|\{x_0\},0) = \frac{1}{N!}\det[K(x_i,x_j;t)]_{i,j=1,\ldots,N}, \tag{A.16}$$

with the kernel function

$$K(x,y;t) = \frac{\mathrm{e}^{-(x-\alpha_0(t)x_0)^2/\alpha_1^2(t)}}{\sqrt{\pi\alpha_1^2(t)}}\sum_{n=0}^{N-1}\frac{1}{2^n n!}H_n\left(\frac{x-\alpha_0(t)x_0}{\alpha_1(t)}\right)H_n\left(\frac{y-\alpha_0(t)x_0}{\alpha_1(t)}\right). \tag{A.17}$$

According to a theorem of RMT [19], which explores the orthogonality of the functions $p_n(x,t)$, it is possible to write the correlation function in a similar factorized form

$$\rho_n(x_1,\ldots,x_n,t) = \det[K(x_i,x_j;t)]_{i,j=1,\ldots,n}. \tag{A.18}$$

The orthogonal and symplectic ensembles with the degenerate initial condition can be treated similarly by means of an anti-orthogonal polynomials method [19]. The correlation functions can also be calculated using well-known theorems of RMT.

A.2. Laguerre ensemble

From table 1 the Laguerre ensemble is characterized by $w(x) = x^\nu \mathrm{e}^{-x}$, $s(x) = x$ and $r(x) = 1+\nu-x$. It is useful to introduce the notation $w_N(\{x\}) = \mathrm{e}^{-\mathrm{Tr}\,X}\prod_i x_i^\nu$, where $X = \mathrm{diag}(x_1,x_2,\ldots,x_N)$. The polynomial part of the eigenfunctions is the normalized Laguerre multivariate polynomials

$$\Phi_\lambda(\{x\}) = \frac{1}{\sqrt{h_\lambda^{(\alpha)}}}L_\lambda^\nu(X,2/\beta), \tag{A.19}$$

where

$$h_\lambda^{(2/\beta)} = \frac{[\nu+q]_\lambda^{(2/\beta)}}{C_N C_\lambda^{(2/\beta)}(1^N)|\lambda|!}; \qquad C_N = \prod_{j=0}^{N-1}\frac{\Gamma(1+\beta/2)}{\Gamma\big(1+(j+1)\beta/2\big)\Gamma(1+\nu+\beta j/2)}, \tag{A.20}$$

with $[a]_\lambda^{(\alpha)} = \prod_{j=1}^N \big(a-\frac{1}{\alpha}(j-1)\big)_{\lambda_j}$ been a generalization of the Pochhammer symbol. The multivariate Laguerre polynomials are defined by Lassale's formula

$$L_\lambda^\nu(X,2/\beta) = \frac{(-1)^{|\lambda|}}{|\lambda|!C_\lambda^{(2/\beta)}(1^N)} \times \exp\left(-\sum_i\left(x_i\frac{\partial^2}{\partial x_i^2}+(1+\nu)\frac{\partial}{\partial x_i}\right)-\beta\sum_{i\neq j}\frac{x_i}{x_i-x_j}\frac{\partial}{\partial x_i}\right)C_\lambda^{(2/\beta)}(X). \tag{A.21}$$

The transition probability is obtained by substituting (A.7) into (A.6). Using a summation identity [29], it can be written in closed form

$$P(\{x\},t|\{x_0\},0) = \frac{P_{st}(X)}{(1-\mathrm{e}^{-t})^{N(\nu+q)}} \exp\left(-\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}(\mathrm{Tr}\,X + \mathrm{Tr}\,X_0)\right) \times {}_0\mathcal{F}_1^{(2/\beta)}\left(\nu+q, \frac{X_0}{1-\mathrm{e}^{-t}}, \frac{\mathrm{e}^{-t}X}{1-\mathrm{e}^{-t}}\right), \tag{A.22}$$

where $P_{st}(X)$ is the stationary solution (11), $q = 1+\beta(N-1)/2$ and

$${}_0\mathcal{F}_1^{(\alpha)}(a;X;Y) = \sum_{\lambda} \frac{1}{|\lambda|![a]_\lambda^{(\alpha)}} \frac{C_\lambda^{(\alpha)}(X)C_\lambda^{(\alpha)}(Y)}{C_\lambda^{(\alpha)}(1^N)} \tag{A.23}$$

is the generalized hypergeometric function. As in the previous example, the problem of finding the n-point correlation functions for general β and an arbitrary initial condition is still open. This job can be done in the particular case of the degenerate initial condition $X_0 = \mathrm{diag}(0,0,\ldots,0)$, where (A.22) can be written in terms of elementary functions

$$\mathcal{P}(\{x\},t|\{0\},0) = \frac{P_{st}(X)}{(1-\mathrm{e}^{-t})^{N(\nu+q)}} \exp\left(-\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\mathrm{Tr}\,X\right). \tag{A.24}$$

To make contact with the classical Laguerre ensemble we rewrite (A.24) explicitly as

$$\mathcal{P}(\{x\},t|\{0\},0) = C_N\alpha^N(t)\prod_{i<j}|\alpha(t)x_i - \alpha(t)x_j|^\beta \prod_{i=1}^N(\alpha(t)x_i)^\nu\,\mathrm{e}^{-\alpha(t)x_i}, \tag{A.25}$$

where $\alpha(t) = (1-\mathrm{e}^{-t})^{-1}$. The correlation function follows immediately from the methods developed for stationary ensembles [19]. In particular, for $\beta = 2$ the orthogonal polynomials method yields the n-point correlation function, with the determinantal structure of equation (A.18), with kernel given by

$$K(x,y;t) = \alpha(t)(\alpha(t)x)^\nu\,\mathrm{e}^{-\alpha(t)x}\sum_{n=0}^{N-1}\frac{n!}{\Gamma(n+\nu+1)}L_n^\nu(\alpha(t)x)L_n^\nu(\alpha(t)y), \tag{A.26}$$

where $L_n^\nu(x)$ is the classical Laguerre polynomial.

To conclude, we emphasize that despite (A.10) and (A.22) represent a solution to the Fokker–Planck equation for general β, they do not yield direct access to the n-point correlation function. In the particular case of the degenerate initial condition the correlation functions can be calculated with an adaptation of the methods of equilibrium ensembles.

Appendix B. Biorthogonal system for polynomial ensembles

We start with the differential equation

$$\left(\frac{\partial}{\partial t} - H_x\right)\chi(x,t) = 0. \tag{B.1}$$

The operator H_x is defined in (29) and has a complete set of integrable eigenfunctions in $[a,b]$ with weight $w(x)$. The general solution of (B.1) is given by

$$\chi(x,t) = \int_a^b \mathrm{d}y\, w(y)g(x,t|y,0)\chi(y,0), \tag{B.2}$$

where $g(x,t|y,t')$ is the one-particle Green's function with spectral decomposition (33). Now consider the equation obtained from (B.1) by changing $t \to -t$

$$\left(\frac{\partial}{\partial t} + H_x\right)\psi(x,t) = 0, \tag{B.3}$$

which correspond to a time reversal operation. The corresponding Green's function $\tilde{g}$, is related to Green's function g through $\tilde{g}(x,t|x',t') = g(x',t'|x,t)$. By imposing the initial conditions

$$\chi_m(x,0) = \frac{\delta(x - x'_{m+1})}{w(x)}, \qquad m = 0, 1, \ldots, N-1 \tag{B.4}$$

$$\psi_n(x,0) = f_n(x), \qquad n = 0, 1, \ldots, N-1, \tag{B.5}$$

we find the following solutions

$$\chi_m(x,t) = g(x,t|x'_{m+1},0), \tag{B.6}$$

$$\psi_n(x,t) = \int_a^b \mathrm{d}y\, w(y) g(y,0|x,t) f_n(y,0). \tag{B.7}$$

We define the overlap matrix

$$F_{n,m}(t) = \int_a^b \mathrm{d}x\, w(x) \psi_n(x,t) \chi_m(x,t). \tag{B.8}$$

Using the definitions of $\psi_n(x,t)$ and $\chi_m(x,t)$ we find

$$F_{n,m}(t) = \int_a^b \mathrm{d}y\, w(y) f_n(y) \int_a^b \mathrm{d}x\, w(x) g(y,0|x,t) g(x,t|x'_{m+1},0). \tag{B.9}$$

The integral in x shows a composition of Green's function from $t = 0$ to an arbitrary time t and goes back to $t = 0$. Using the composition rule

$$\int_a^b \mathrm{d}x\, w(x) g(y,0|x,t) g(x,t|x'_{m+1},0) = g(y,0|x'_{m+1},0) = \frac{\delta(y - x'_{m+1})}{w(y)}, \tag{B.10}$$

we see that the overlap matrix is time independent and depends only on the choice of the initial condition $f(x)$

$$F_{n,m}(t) = F_{n,m} = f_n(x'_{m+1}). \tag{B.11}$$

Choosing $f_n(x)$ to be the Lagrange interpolation polynomial

$$f_n(x) = L_{n+1}(x) = \prod_{\substack{l=0 \\ l \neq n}}^{N-1} \frac{x - x'_{l+1}}{x'_{n+1} - x'_{l+1}}, \qquad n = 0, \ldots, N-1, \tag{B.12}$$

we diagonalize the overlap matrix

$$F_{n,m} = \delta_{n,m}. \tag{B.13}$$

Therefore, the set of functions $\psi_n(x,t)$ e $\chi_m(x,t)$ form a biorthogonal system.

$$\int_a^b \mathrm{d}x\, w(x) \psi_n(x,t) \chi_m(x,t) = \delta_{n,m}. \tag{B.14}$$

Appendix C. Biorthogonal system for transfer matrix ensembles

The central idea is to perform a linear combination of the matrix elements

$$\phi_n(x,t) \to \psi_n(x,t) = \sum_{l=0}^{\infty} \alpha_{ln} \phi_l(x,t), \tag{C.1}$$

and look for the appropriate coefficients in order to diagonalize the new overlap matrix

$$\int_1^\infty \mathrm{d}x\, w(x)\psi_n(x,t)\chi_m(x,t) = \delta_{n,m}. \tag{C.2}$$

Applying the transformation (C.1) into (C.2) and using (141) we find

$$\sum_{l=0}^{N-1} \alpha_{ln}\phi_l(x'_{m+1}) = \delta_{n,m}. \tag{C.3}$$

By definition the Lagrange polynomial

$$L_{n+1}(x) = \prod_{\substack{l=0\\ l\neq n}}^{N-1} \frac{x - x'_{l+1}}{x'_{n+1} - x'_{l+1}}, \qquad n = 0, \ldots, N-1, \tag{C.4}$$

interpolates the function $\phi_k(x)$ at points x'_{m+1}, and we obtain

$$\phi_k(x) = \sum_{m=0}^{N-1} \phi_k(x'_{m+1}) L_{m+1}(x). \tag{C.5}$$

Therefore, from (C.3) and (C.5) we can write

$$L_{n+1}(x) = \sum_{l=0}^{N-1} \alpha_{ln}\phi_l(x). \tag{C.6}$$

The coefficients α_{ln} can be found by using the orthogonality condition of the Jacobi polynomials $\phi_l(x)$, yielding

$$\alpha_{nm} = \int_{-1}^{1} \mathrm{d}x\, v(x)\phi_n(x)L_{m+1}(x). \tag{C.7}$$

Thus the transformation that diagonalizes the overlap function is

$$\phi_n(x,t) \to \psi_n(x,t) = \int_{-1}^{1} \mathrm{d}y\, v(y)L_{n+1}(y) \sum_{l=0}^{N-1} \phi_l(y)\phi_l(x,t). \tag{C.8}$$

This equation can be put in a more compact form by introducing Green's function defined by the equation

$$\left(\frac{\partial}{\partial t} - H\right) g_2(x,t|x',t') = 0; \qquad g_2(x,t|y,t) = \frac{\delta(x-y)}{v(x)}. \tag{C.9}$$

Since the eigenfunctions of H are Jacobi polynomials, Green's function has the following discrete spectral decomposition

$$g_2(x,t|y,t') = \sum_{n=0}^{\infty} \mathrm{e}^{-\epsilon_n(t-t')}\phi_n(x)\phi_n(y). \tag{C.10}$$

As L_{n+1} is a polynomial of degree $N-1$, we can use the orthogonality condition

$$\int_{-1}^{1} \mathrm{d}y\, v(y)L_{n+1}(y)\phi_l(y) = 0, \qquad l > N-1, \tag{C.11}$$

to extend the upper limit of the sum in (C.8) to infinity, which by definition gives Green's function. Therefore, we write the transformation in the following compact form

$$\psi_n(x,t) = \int_{-1}^{1} \mathrm{d}y\, v(y)L_{n+1}(y)g_2(x,t|y,0). \tag{C.12}$$

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